SCIENTIFIC PROGRAMME

Tuesday, May 30 **Ostromecko Palace**

Chair: Małgorzata Filipczak

Chair: Marek Balcerzak

Wednesday, May 31 **Copernicanum, Bydgoszcz**

10:00–10:30 addresses

Chair: Jacek Jędrzejewski

10:30–11:10 Tomasz Natkaniec (Gdańsk):
Professor Zbigniew Grande's contribution to the Real Functions Theory

11:10–11:40 coffee break

Thursday, June 1 **Ostromecko Palace**

Chair: Attila Gilányi

Chair: Valentin A. Skvortsov

The Baire category of ideal convergent subseries and rearrangements of a divergent series

Marek Balcerzak (Łódź)

Let $\mathcal{I}$ be a 1-shift-invariant ideal on $\mathbb{N}$ with the Baire property. Assume that a series $\sum_n x_n$ with terms in a real Banach space X is not unconditionally convergent. We show that the sets of $\mathcal{I}$-convergent subseries and of $\mathcal{I}$-convergent rearrangements of a given series are meager in the respective Polish spaces. A stronger result, dealing with $\mathcal{I}$-bounded partial sums of a series, is obtained if X is finite-dimensional. We apply the main theorem to series of functions with the Baire property, from a Polish space to a separable Banach space over $\mathbb{R}$, under the assumption that the ideal $\mathcal{I}$ is analytic or coanalytic.

(These are results obtained together with Michał Popławski and Artur Wachowicz.)

Institute of Mathematics, Łódź University of Technology
Wólczańska 215, 93-005 Łódź, Poland
`marek.balcerzak@p.lodz.pl`

On generating regular Cantorvals connected with geometric Cantor sets

Artur Bartoszewicz (Łódź)

We show that the Cantorvals connected with the q-Cantor sets are not achievement sets of any series. However some of them are attractors of IFS consisting of affine functions.

(Joint results with Małgorzata Filipczak, Szymon Głąb, Franciszek Prus-Wiśniowski and Jarosław Swaczyna.)

Institute of Mathematics, Łódź University of Technology
Wólczańska 215, 93-005 Łódź, Poland
artur.bartoszewicz@p.lodz.pl

Decomposition theorems using strong generalized derivatives

ZOLTÁN BOROS (Debrecen)

In this presentation we consider a sharpening of the decomposition theorem for strongly $\mathbb{Q}$-differentiable functions [1], and its counterpart in the theory of convexity. Our investigations are motivated by a related paper of Broszka and Grande [2] as well as by Ng's celebrated decomposition theorem for Wright-convex functions [3].

Let I denote an open interval in the real line, and let us consider a function $f\colon I \to \mathbb{R}$. For $x \in I$ and $h \in \mathbb{R}$, we define the lower and upper strong binary derivatives of f by

$$\underline{D}_h^{\diamond} f(x) = \liminf_{\substack{y \to x \\ n \to \infty}} 2^n \big(f(y + 2^{-n} h) - f(y) \big)$$

and

$$\overline{D}_h^{\diamond} f(x) = \limsup_{\substack{y \to x \\ n \to \infty}} 2^n \big(f(y + 2^{-n} h) - f(y) \big),$$

respectively. We call f strongly binary differentiable if

$$\underline{D}_h^{\diamond} f(x) = \overline{D}_h^{\diamond} f(x) \in \mathbb{R}$$

holds for every $x \in I$ and $h \in \mathbb{R}$. We say that f has increasing strong binary derivatives if

$$-\infty < \overline{D}_h^{\diamond} f(x) \leq \underline{D}_h^{\diamond} f(y) < +\infty$$

holds for every $h > 0$ and $x, y \in I$ such that $x < y$. These properties are characterized by the following decomposition theorems:

Theorem 1. *The function f is strongly binary differentiable if, and only if, there exist a continuously differentiable function $g\colon I \to \mathbb{R}$ and an additive mapping $\varphi\colon \mathbb{R} \to \mathbb{R}$ such that $f(x) = g(x) + \varphi(x)$ for every $x \in I$.*

Theorem 2. *The function f has increasing strong binary derivatives if, and only if, there exist a convex function $g\colon I \to \mathbb{R}$ and an additive mapping $\varphi\colon \mathbb{R} \to \mathbb{R}$ such that $f(x) = g(x) + \varphi(x)$ for every $x \in I$.*

Applying these results, we characterize affine (respectively, Wright-convex) functions as locally approximately affine (respectively, locally approximately Wright-convex) functions in a specific sense. In particular, we obtain a localization principle for these classes of functions.

References

[1] Z. Boros, *Strongly $\mathbb{Q}$-differentiable functions*, Real Analysis Exchange, **27**(1) (2001/2002), 17–25.

[2] D. Broszka, Z. Grande, *On $\mathcal{I}$-differentiation*, Tatra Mountains Math. Publ., **35** (2007), 25–40.

[3] C. T. Ng, *Functions generating Schur-convex sums*, In: General Inequalities, **5** (Oberwolfach, 1986), Internat. Schriftenreihe Numer. Math., **80**, Birkhäuser, Basel, 1987, 433–438.

Institute of Mathematics, University of Debrecen, Hungary
zboros@science.unideb.hu

Achievement sets of absolutely summable sequences

Małgorzata Filipczak (Łódź)

Let $(x_n) = (x_1, x_2, \ldots)$ be a sequence tending to zero. We will investigate the set of all subsums of the series $\sum_{n=1}^{\infty} x_n$,

$$E(x_n) = \left\{ \sum_{n=1}^{\infty} \varepsilon_n x_n : (\varepsilon_n) \in \{0,1\}^{\mathbb{N}} \right\},$$

called the *achievement set of the sequence* (x_n).

We will focus on monotone, absolutely summable sequences of positive terms for which the achievement sets are neither finite unions of intervals, nor nowhere dense sets. Particular attention will be paid to achievement sets of so-called *multigeometric sequences*.

This leads us to study topological and measure properties of the fractal $K = K(\Sigma; q)$, which is the unique solution of the equation $K = \Sigma + qK$ for a given finite set $\Sigma \subset \mathbb{N}$ and a positive real number $q < 1$.

Faculty of Mathematics and Computer Science, Łódź University
Banacha 22, 90-238 Łódź, Poland
`malfil@math.uni.lodz.pl`

On higher-order and strongly convex functions

ATTILA GILÁNYI (Debrecen)

It is well-known that a real valued function f defined on an interval I is called convex if it satisfies the inequality

$$f(tx + (1-t)y) \le tf(x) + (1-t)f(y)$$

for all $x, y \in I$ and $t \in [0,1]$, it is said to be Jensen-convex if it fulfils

$$f\left(\frac{x+y}{2}\right) \le \frac{f(x)+f(y)}{2}$$

for all $x, y \in I$.

In this talk, we investigate various generalizations of the concepts above. In particular, based on the corresponding definitions of E. Hopf (1926), T. Popoviciu (1934) and B. T. Polyak (1966), we consider strongly convex, strongly Wright-convex and strongly Jensen-convex functions of higher order, we present characterization theorems for them via (generalized) derivatives and we prove that the properties studied are localizable.

University of Debrecen, Hungary
gilanyi.attila@inf.unideb.hu

A stronger version of the Hadamard inequality

Edyta Hetmaniok, Michał Różański, Marcin Szweda, Roman Wituła (Gliwice)

In 1893 Jacques Hadamard presented the following inequality connected with the determinant of Gram matrix in the unitary space

$$\Gamma(x_1,\dots,x_n)=\det\begin{pmatrix}\langle x_1|x_1\rangle & \cdots & \langle x_1|x_n\rangle\\ \vdots & \ddots & \vdots\\ \langle x_n|x_1\rangle & \cdots & \langle x_n|x_n\rangle\end{pmatrix}\le\prod_{i=1}^{n}\|x_i\|^2.$$

The aim of our presentation is to introduce a few generalizations of this inequality. Among others, we have obtained the inequality of the form

$$\Gamma(x_1,\dots,x_n)+\left(|\langle x_m|x_1\rangle|\prod_{i=1}^{m-1}|\langle x_i|x_{i+1}\rangle|\right)\left(|\langle x_n|x_{m+1}\rangle|\prod_{i=m+1}^{n-1}|\langle x_i|x_{i+1}\rangle|\right)\le\prod_{i=1}^{n}\|x_i\|^2.$$

An additional advantage of our research is the derivation of the following asymptotic relation for the normalized vectors $x_1,\dots,x_n$ of the given unitary space

$$\lim_{\delta\to 0}\frac{1-\Gamma(x_1,\dots,x_n)}{\delta}=1,$$

where $\delta=\sum_{1\le i<j\le n}|\langle x_i|x_j\rangle|^2$.

Silesian University of Technology, Faculty of Applied Mathematics
Kaszubska 23, 44-101 Gliwice, Poland
Roman.Witula@polsl.pl

On conditional polynomial functional equations

JUDIT KOSZTUR (Debrecen)

A function f defined on the real line $\mathbb{R}$ mapping into a linear space Y is called a polynomial function of degree n if it satisfies the equation

$$\Delta_y^{n+1} f(x) = 0$$

for all $x, y \in \mathbb{R}$, where n is a fixed non-negative integer and Δ is the difference operator defined by

$$\Delta_y^1 f(x) = f(x+y) - f(x) \qquad (x, y \in \mathbb{R})$$

and, for a positive integer n,

$$\Delta_y^{n+1} f(x) = \Delta_y^1 \Delta_y^n f(x) \qquad (x, y \in \mathbb{R}).$$

In this talk, we consider the situation, when the equation above is valid for some special elements $x, y \in \mathbb{R}$ only.

(Joint work with Katarzyna Chmielewska and Attila Gilányi.)

University of Debrecen, Hungary
`kosztur.judit@science.unideb.hu`

On the Adams metric in the space of continuous functions of bounded variation

Stanisław Kowalczyk (Słupsk)

Let $C[0,1]$ be the space of all continuous real-valued functions defined on $[0,1]$ with the supremum norm $\|f\| = \sup_{t\in[0,1]} |f(t)|$. There are some natural operations on $C[0,1]$, for example, addition, multiplication, minimum and maximum. In [7, 14, 15] such operations were investigated. All the operations are continuous but only addition, minimum and maximum are open as mappings from $C[0,1]\times C[0,1]$ to $C[0,1]$.

Remark (Fremlin's example). [7] In 2004, D. H. Fremlin observed that for $f\colon [0,1]\to\mathbb{R}$, $f(x) = x-\frac{1}{2}$, one has $f^2 \in B^2\big(f,\frac{1}{2}\big)\setminus \operatorname{Int} B^2\big(f,\frac{1}{2}\big)$. Hence multiplication is not an open mapping from $C[0,1]\times C[0,1]$ into $C[0,1]$.

Definition 1. [5, 7] *A map between topological spaces is weakly open if the image of every non-empty open set has a non-empty interior.*

In [7] it is shown that the multiplication in $C[0,1]$ is a weakly open operation. This was generalized in [11] for $C(0,1)$ and in [5] for $C(X)$, where X is an arbitrary interval. In [10, 11] there are considered some properties of multiplication and other operations in the algebra $C(X)$ of real-valued continuous functions defined on a compact topological space X. Properties of the product of open balls and of n open sets in the space of continuous functions on $[0,1]$ are studied in [8] and [9], respectively. There is an increasing interest in the study of concepts related to the openness and weak openness of natural bilinear maps on certain function spaces. The reason of it may be that the classical Banach open mapping principle is not true for bilinear maps. In [4], the authors show that multiplication from $L^p(X)\times L^q(X)$ onto $L^1(X)$, where (X,μ) is an arbitrary measure space and $\frac{1}{p}+\frac{1}{q}=1$, $1\le p,q\le\infty$, is an open mapping.

We study a problem, stated in [6], concerning openness and weak openness in the space $CBV[0,1]$ of functions of bounded variation and in the space $CBV[0,1]$ of continuous functions of bounded variation, both defined on $[0,1]$. There are two natural norms in $BV[0,1]$ and $CBV[0,1]$: the supremum norm $\|f\| = \sup_{t\in[0,1]} |f(t)|$ and the norm defined by variation, $\|f\|_{BV} = |f(0)| + V_0^1(f)$, where

$$V_a^b(f) = \sup_{a=t_0<t_1<\ldots<t_n=b} \sum_{i=1}^{n} |f(t_i)-f(t_{i-1})|$$

is the variation of f on $[a,b]$. It is worth to mention that $(BV[0,1],\|\ \|_{BV})$ and $(CBV[0,1],\|\ \|_{BV})$ are complete, whereas $(BV,\|\ \|)$ and $(CBV,\|\ \|)$ are not. In [13] some answers are presented. In [6] there were stated questions concerning weak openness of multiplication in the space $CBV[0,1]$ with so-called *Adams metric.*

Definition 2. *The Adams metric in $CBV[0,1]$ is defined by*

$$\varrho_A(f,g) = \int_0^1 |f(x)-g(x)|\,dx + \left|V_0^1(f)-V_0^1(g)\right|.$$

On first sight this metric is quite strange (in $BV[0,1]$, ϱ_A is only a pseudometric), but convergence in ϱ_A is equivalent to so-called convergence in variation (see [1]). Moreover, convergence in Adams metric is weaker than convergence in the norm defined by variation, $\|f\|_{BV} = |f(0)| + V_0^1(f)$.

Theorem 3. *If a sequence $(f_n)_{n\in\mathbb{N}}$ of functions from $CBV[0,1]$ is convergent to $f\in CBV[0,1]$ in Adams metric then it is uniformly convergent.*

Thus convergence in Adams metric is stronger than convergence in supremum norm $\|\ \|$ and it is weaker then convergence in the norm $\|\ \|_{BV}$.

Here are the most important results we obtained.

Theorem 4. *Multiplication is a weakly open operation in the space $(CBV[0,1],\varrho_A)$.*

Theorem 5. *Addition* $+\colon (CBV[0,1],\varrho_A)\times(CBV[0,1],\varrho_A)\to(CBV[0,1],\varrho_A)$ *is continuous at* (f,g) *if and only if* $V_0^1(f+g)=V_0^1(f)+V_0^1(g)$.

Theorem 6. *Let* $f,g\in CBV[0,1]$. *Then multiplication in* $(CBV[0,1],\varrho_A)$ *is continuous at* (f,g) *if and only if* $V_0^1(|f|+|g|)=V_0^1(|f|)+V_0^1(|g|)$.

Thus the sets of continuity of addition and multiplication in $(CBV[0,1],\varrho_A)$ are incomparable.

References

[1] C. R. Adams, *The space of functions of bounded variation and certain general spaces*, Trans. Amer. Math. Soc., **42** (1937), 421–438.
[2] C. R. Adams, J. A. Clarkson, *On convergence in variation*, Bull. Amer. Math. Soc., **10** (1934), 413–417.
[3] C. R. Adams, A. P. Morse, *On the space (BV)*, Trans. Amer. Math. Soc., **42** (1937), 194–205.
[4] M. Balcerzak, A. Majchrzycki, A. Wachowicz, *Openness of multiplication in some function spaces*, Taiwanese J. Math., **17** (2013), 1115–1126.
[5] M. Balcerzak, A. Maliszewski, *On multiplication in spaces of continuous functions*, Colloq. Math., **122** (2011), 247–253.
[6] M. Balcerzak, F. Strobin, A. Wachowicz, *Billinear mappings-selected properties and problems*, In: Traditional and Present-Day Topics in Real Analysis, University of Łódź, 2013, § 18.
[7] M. Balcerzak, A. Wachowicz, W. Wilczyński, *Multiplying balls in the spaces of continuous functions on* $[0,1]$, Studia Math., **170** (2005), 203–209.
[8] E. Behrends, *Walk the dog, or: products of open balls in the space of continuous functions*, Func. Approx. Comment. Math., **44** (2011), 153–164.
[9] E. Behrends, *Products of* n *open subsets in the space of continuous functions on* $[0,1]$, Studia Math., **204** (2011), 73–95.
[10] A. Komisarski, *A connection between multiplication in* $C(X)$ *and the dimension of* X, Fund. Math., **189** (2006), 149–154.
[11] S. Kowalczyk, *Weak openness of multiplication in* $C(0,1)$, Real Anal. Exchange, **35** (2010), 235–241.
[12] S. Kowalczyk, *On operations in* $C(X)$ *determined by continuous functions*, Acta Math. Hungar., **142** (2013), 56–71.
[13] S. Kowalczyk, *Openness and weak openness of multiplication in the space of functions of bounded variation*, J. Math. Anal. Appl., **434** (2016), 1290–1301.
[14] A. Wachowicz, *Baire category and standard operations on pairs of continuous functions*, Tatra Mt. Math. Publ., **24** (2002), 141–146.
[15] A. Wachowicz, *On some residual set*, PhD dissertation, Łódź University of Technology, 2004 (in Polish).

Institute of Mathematics, Pomeranian Academy in Słupsk
Kozietulskiego 6-7, 76-200 Słupsk, Poland
stanislaw.kowalczyk@apsl.edu.pl

On approximately continuous functions attracting positive entropy at a point

Anna Loranty and Ryszard J. Pawlak (Łódź)

During the talk we will concentrate on approximately continuous functions $f\colon \mathbb{I} \to \mathbb{I}$, where $\mathbb{I} = [0,1]$.

We shall say that a function $f\colon \mathbb{I} \to \mathbb{I}$ is *approximately continuous at a point* $x_0 \in \mathbb{I}$ if there is a Lebesgue measurable set $A \subset I$ such that x_0 is a density point of A and $\lim_{A \ni x \to x_0} f(x) = f(x_0)$. Clearly, if $x_0 = 0$ ($x_0 = 1$) then x_0 is a right (left) density point of A. Moreover, if we additionally assume that an entropy of f on A is equal to 0 then we obtain a 0-*approximately continuous function at a point* x_0 which will play a special role in our considerations.

It is commonly accepted that if the entropy is positive, the function is chaotic. The analysis of different examples of functions leads us to the interesting observation that chaos, and thereby entropy of a function, may be focused around one point. There are a lot of theories, which emphasize the importance of the problem connected with focusing the chaos on a set or at a point. During the talk apart from a classical entropy of a function on a set we will consider also an entropy of a function on a set with respect to a family of closed subsets of $\mathbb{I}$.

Let $\mathcal{F}$ be the family of all closed sets in $\mathbb{I}$. By $\mathcal{F}_c^Y$, where $Y \subset \mathbb{I}$, we will denote the family of all finite sequences of sets from $\mathcal{F}$ consisting of pairwise disjoint sets which are contained in Y. If $F = (A_1, \ldots, A_m) \in \mathcal{F}_c^Y$ and $f\colon \mathbb{I} \to \mathbb{I}$ is a function then we define a structural matrix $\mathcal{M}_{F,f} = [a_{ij}]_{i,j=1}^m$ in the following way: $a_{ij} = 1$ if $A_j \subset f(A_i)$ and $a_{ij} = 0$ otherwise. A *generalized entropy of f with respect to the sequence* $F \in \mathcal{F}_c^Y$ is the number $H_f(F) = \log \sigma(\mathcal{M}_{F,f})$ if $\sigma(\mathcal{M}_{F,f}) > 0$ and $H_f(F) = 0$ if $\sigma(\mathcal{M}_{F,f}) = 0$, where

$$\sigma(\mathcal{M}_{F,f}) = \limsup_{n\to\infty} \sqrt[n]{\operatorname{tr}(\mathcal{M}_{F,f}^n)}.$$

An *entropy of f on Y with respect to the family $\mathcal{F}$* is the number

$$H_f(Y) = \sup\left\{\frac{1}{n} H_{f^n}(F) : F \in \mathcal{F}_c^Y\right\}.$$

We shall say that a Darboux function $f\colon \mathbb{I} \to \mathbb{I}$ *attracts positive entropy at point* $x_0 \in \mathbb{I}$ if for any $\varepsilon > 0$ there is $\delta > 0$ such that for any Darboux function $g \in B(f,\delta)$ we have $H_g(B(x_0,\varepsilon)) > 0$ (where $B(x_0,\varepsilon)$ is a ball of radius ε and a center x_0).

The results presented in our lecture will be connected, among others, with existence of a function g lying "near" a function f and attracting a positive entropy at a fixed point of f. In addition, we will require that the constructed functions have additional properties for example we will require that they are 0-approximately continuous at this point.

Faculty of Mathematics and Computer Science, Łódź University
Banacha 22, 90-238 Łódź, Poland
`loranta@math.uni.lodz.pl`, `rpawlak@math.uni.lodz.pl`

Achievement sets of conditionally convergent series

Jacek Marchwicki, Szymon Głąb, and Artur Bartoszewicz (Łódź)

We call the set $A(x_n) = \left\{\sum_{n=1}^{\infty} \varepsilon_n x_n : (\varepsilon_n) \in \{0,1\}^{\mathbb{N}}\right\}$ the *set of subsums* or the *achievement set*. By $SR(x_n) = \{\sum_{n=1}^{\infty} x_{\sigma(n)} : \sigma \in S_\infty\}$ we denote the sum range. Due to Guthrie, Nymann and Saenz we know that the achievement set of an absolutely summable sequence of reals can be a finite set, a finite union of intervals, homeomorphic to the Cantor set or it can be a so called Cantorval. A Cantorval is a set homeomorphic to the union of the Cantor set and sets which are removed from the unite segment by even steps of the Cantor set construction.

Theorem 3. *For sequences of reals with* $\lim\limits_{n\to\infty} x_n = 0$ *we have:*

- *A series* $\sum_{n=1}^{\infty} x_n$ *is potentially conditionally convergent (both series of positive and negative terms are divergent) if and only if* $A(x_n) = \mathbb{R}$*.*
- *A series of negative terms is convergent and a series of positive terms is divergent (or vice versa) if and only if the achievement set of* (x_n) *is a half line.*

Considering the sets of subsums of series (or achievement sets) we show that for conditionally convergent series the multidimensional case is much more complicated than that of the real line and we are still far from the full topological classification of such sets. Many surprising examples are presented and the ideas standing behind them are catched in general theorems. We observe among others that for the achievement set $A(x_n)$ of conditionally convergent series in $\mathbb{R}^2$ the following are possible.

- The intersection of $A(x_n)$ and $SR(x_n)$ could be a singleton and moreover we mention that it is always nonempty set;
- $A(x_n)$ can be a graph of function;
- $A(x_n)$ can be a dense set in $\mathbb{R}^2$ with an empty interior;
- $A(x_n)$ can be neither F_σ nor G_δ-set;
- $A(x_n)$ can be an open set not equal to the whole $\mathbb{R}^2$.

Institute of Mathematics, Łódź University of Technology
Wólczańska 215, 93-005 Łódź, Poland
marchewajaclaw@gmail.com, szymon.glab@p.lodz.pl, arturbar@p.lodz.pl

Approximation of real-valued non-additive measures

Oleh Nykyforchyn (Bydgoszcz), Inna Hlushak (Ivano-Frankivsk)

Capacities, or non-additive measures were introduced by Choquet [1] and found numerous applications in different theories. Spaces of upper semicontinuous capacities on compacta were systematically studied in [2]. In particular, in the latter paper functoriality of the construction of a space of capacities was proved and Prokhorov-style and Kantorovich–Rubinstein style metrics on the set of capacities on a metric compactum were introduced. It turned out that not all metrics for *additive* measures are meaningful for *non-additive* ones, hence in the sequel we consider the Prokhorov-style metric as reflecting the best our intuitive notion of one real-valued set function being close to another. Needs of practice require that a capacity can be approximated with capacities of simpler structure or with some convenient properties.

We follow the terminology and notation of [2] and denote by $\exp X$ the set of all non-empty closed subsets of a compactum X. We call a function $c\colon \exp X \cup \{\emptyset\} \to I$ a *capacity* on a compactum X if the three following properties hold for all subsets $F, G \underset{\mathrm{cl}}{\subset} X$:

(1) $c(\emptyset) = 0$;
(2) if $F \subset G$, then $c(F) \le c(G)$ (monotonicity);
(3) if $c(F) < a$, then there is an open subset $U \supset F$ such that for all $G \subset U$ the inequality $c(G) < a$ is valid (upper semicontinuity).

If, additionally, $c(X) = 1$ (or $c(X) \le 1$) holds, then the capacity is called *normalized* (resp. *subnormalized*). We denote by MX and $\underline{M}X$ the sets of all normalized and of all subnormalized capacities respectively. We consider a metric on the set $\underline{M}X$ of subnormalized capacities on a metric compactum (X, d):

$$\hat{d}(c, c') = \inf\left\{\varepsilon > 0 \mid c(\bar{O}_\varepsilon(F)) + \varepsilon \ge c'(F),\ c'(\bar{O}_\varepsilon(F)) + \varepsilon \ge c(F),\ \forall F \underset{\mathrm{cl}}{\subset} X\right\},$$

here $\bar{O}_\varepsilon(F)$ is the closed ε-neighborhood of a subset $F \subset X$. This metric determines a compact topology on $\underline{M}X$ [2]. We consider the following subclasses of MX:

1) $M_\cap X$ is the set of the so-called $\cap$-*capacities* (or necessity measures) with the property: $c(A \cap B) = \min\{c(A), c(B)\}$ for all $A, B \underset{\mathrm{cl}}{\subset} X$;

2) $M_\cup X$ is the set of the so-called $\cup$-*capacities* (or possibility measures) with the property: $c(A \cup B) = \max\{c(A), c(B)\}$ for all $A, B \underset{\mathrm{cl}}{\subset} X$;

3) class MX_0 of capacities defined *on a closed subspace* $X_0 \subset X$; we regard each capacity c_0 on X_0 as a capacity on X extended with the formula $c(F) = c_0(F \cap X_0)$, $F \underset{\mathrm{cl}}{\subset} X$;

4) class $M_{Lip}X$ of capacities that are non-expanding w.r.t. the Hausdorff metric on $\exp X$; *and, finally, the most important*

5) class PS of *additive* normalized regular measures.

Analogous subclasses are defined in $\underline{M}X$, with the obvious denotations.

We will present methods for optimal approximation of an arbitrary non-additive measure with a capacity from the above classes w.r.t. the Prokhorov-style metric.

References

[1] G. Choquet, *Theory of Capacity*, Ann. l'Institute Fourier, **5** (1953-54), 131–295.
[2] M.M. Zarichnyi, O.R. Nykyforchyn, *Capacity functor in the category of compacta*, Sbornik Mathematics, **199**(2) (2008), 159–184.

Institute of Mathematics, Kazimierz Wielki University
Pl. Weyssenhoffa 11, 85-072 Bydgoszcz, Poland
oleh.nyk@gmail.com

Vasyl' Stefanyk Precarpathian National University
Ivano-Frankivsk, Ukraine
inna_gl@rambler.ru

On almost everywhere convergence of multiple Fourier–Haar series

Giorgi Oniani (Kutaisi)

We will discuss optimal conditions guaranteeing an almost everywhere convergence of multiple Fourier–Haar series by spheres and generally by homothetic copies of a fixed convex set.

Akaki Tsereteli State University, Department of Mathematics
59 Tamar Mepe St., Kutaisi 4600, Georgia
oniani@atsu.edu.ge

Uniqueness for the Vilenkin systems and problems of recovering of mixed type for summable functions

Mikhail G. Plotnikov (Vologda)

Let $\mathcal{P}$ be any sequence of prime numbers, $G_{\mathcal{P}}$ be the $\mathcal{P}$-adic group, $\{\gamma_n\}$ be the Vilenkin–Paley system on $G_{\mathcal{P}}$, and μ be the normalized Haar measure on $G_{\mathcal{P}}$ (see, for example, [1, 2]). The most known example of such a group is the Cantor dyadic group for which the system $\{\gamma_n\}$ is the classical Walsh system [2, 5].

Sets of uniqueness for the Vilenkin systems are studied in some works (see, for instance, [3, 4]). We consider rearrangements of the Vilenkin systems. The following result is true: *for any $\delta > 0$ there exist rearranged Vilenkin systems $\left\{\gamma_{T(n)}\right\}$ having closed sets of uniqueness A with $\mu(A) > 1-\delta$ and such that*

$$\lim_{N\to\infty} \frac{\#\left\{n \in \mathbb{N}\colon\ (n < N)\ \&\ (T(n) = n)\right\}}{N} = 1.$$

The next statement is a recovering theorem of mixed type for summable functions: *let $\mathbf{A}$ be the class of sets A, and $\mathbf{T}$ be the family of rearrangements T, mentioned above. Consider a set $A \in \mathbf{A}$, a rearrangement $T \in \mathbf{T}$, a function $f \in L\left(G_{\mathcal{P}}\right)$, and the series S being the Fourier series of f with respect to the rearranged Vilenkin system $\left\{\gamma_{T(n)}\right\}$. Suppose the series S converges to a finite sum at every "$\mathcal{P}$-adic rational" point $g \in G_{\mathcal{P}} \setminus A$. Then there exists an effective procedure for recovering μ-almost all values of the function f on the group $G_{\mathcal{P}}$ from values of f on the set $G_{\mathcal{P}} \setminus A$.*

References

[1] G. N. Agaev, N. Ya. Vilenkin, G. M. Dzhafarli, A. I. Rubinstein, *Multiplicative system of functions and harmonic analysis on zero-dimensional groups*, Baku, 1981 (in Russian).

[2] B. I. Golubov, A. V. Efimov, V. A. Skvortsov, *Walsh Series and Transforms. Theory and Applications*, Kluwer Academic Publishers, 1991.

[3] D. C. Harris, *Sets of uniqueness and closed subgroups in Vilenkin groups*, Anal. Math., **16**(2) (1990), 115–122.

[4] N. Kholshchevnikova, V. Skvortsov, *On U- and M-sets for series with respect to characters of compact zero-dimensional groups*, J. Math. Anal. Appl., **446**(1) (2017), 383–394.

[5] F. Schipp, W. R. Wade, P. Simon, *Walsh Series. An Introduction to Dyadic Harmonic Analysis*, Adam Hilger Publ., Bristol, New York, 1990.

Vologda Vereshchagin Academy
Schmidt St. 2, Molochnoe, Vologda 160555, Russia
mgplotnikov@gmail.com

Linear extensions of some Baire functions with values in vector spaces

Waldemar Sieg (Bydgoszcz)

Let X be the Hausdorff topological space and let Y be the separable Fréchet space. Let also $\omega = \mathbb{R}^{\mathbb{N}}$ be the space of all countable sequences with real values considered with the product topology. Let $\mathcal{C}(X, \mathbb{R})$ denote the set of all real continuous functions on X. The set of all real functions with a closed graph is denoted by $\mathcal{U}(X, \mathbb{R})$. Obviously $\mathcal{C}(X, \mathbb{R}) \subset \mathcal{U}(X, \mathbb{R})$. By $\mathcal{F}$ we denote a certain property of some family of functions defined on X (i.e., continuity, closedness of the graph, etc.). Put $\mathcal{F}(X, Y) = \{f\colon X \to Y : f \text{ has the property } \mathcal{F} \text{ on } X\}$. Let A be a nonempty subset of X considered with the topology induced from X and let $\mathcal{F}(A, Y)$ be a set of functions defined on A with the same meaning as $\mathcal{F}(X, Y)$.

I consider the Borsuk–Dugundji type of extension theorem for piecewise continuous and closed graph functions defined on A and with values in ω or Y. In 2010 we proved [6] that if X is a P-space (i.e., every G_δ-subset of X is open) then $\mathcal{C}(X, \mathbb{R}) = \mathcal{U}(X, \mathbb{R})$ and thus (formally) for every closed subset A of X, every $f_0 \in \mathcal{U}(A, \mathbb{R})$ can be extended to $f \in \mathcal{U}(X, \mathbb{R})$. This observation has led us to the conjecture that a Tietze-type theorem should hold for the class of closed graph functions defined on some subsets of the Hausdorff space X and with values in ω or Y. The conjecture is confirmed in one of our main results, where we've showed that there is an extension operator from $\mathcal{U}(A, \omega)$ into $\mathcal{U}(X, \omega)$.

The function f is *piecewise continuous* if there is a sequence (W_n) of closed subsets of X such that $X = \bigcup_{n=1}^{\infty} W_n$ and all the restrictions $f\restriction_{W_n}$ are continuous. Since the sum of two closed sets is closed, we can assume that the sequence (W_n) is upward. In the second part of the lecture I will talk about an analogous linear extension operator for piecewise continuous functions with values in ω or Y.

References

[1] K. Borsuk, *Über Isomorphic der Funktionalräume*, Bull. Int. Acad. Polon. Sci., 1933, 1–10.
[2] J. Dugundji, *An extension of Tietze's theorem*, Pacific J. Math., **1** (1951), 353–367.
[3] O. F. K. Kalenda, J. Spurný, *Extending Baire-one functions on topological spaces*, Topology Appl., **149**(1-3) (2005), 195–216.
[4] K. Kuratowski, *Sur les théorémes topologiques de la théorie des fonctions de variables réelles*, C. R. Acad. Sci. Paris, **197** (1933), 19–20.
[5] M. Wójtowicz, W. Sieg, *Affine extensions of functions with a closed graph*, Opuscula Math., **35**(6) (2015), 973–978.
[6] M. Wójtowicz, W. Sieg, *P-spaces and an uncoditional closed graph theorem*, RACSAM, **104**(1) (2010), 13–18.

Institute of Mathematics, Kazimierz Wielki University
Pl. Weyssenhoffa 11, 85-072 Bydgoszcz, Poland
`waldeks@ukw.edu.pl`

On some symmetric derivatives and integrals

Valentin A. Skvortsov (Moscow)

We present a survey of results related to properties and interrelation of some Perron type integrals defined by various symmetric derivates and solving the problem of recovering the coefficients of convergent trigonometric series by generalized Fourier formulas. A special emphasis is given to classes of integrable functions with continuous primitives. In particular we show that the Marcinkiewicz–Zygmund integral defined by symmetric Borel derivative remains to be essentially more general than Burkill's SCP-integral and James' P^2-integral on such a class of integrable functions.

Moscow State University
Moscow, Russia
`vaskvor2000@yahoo.com`

On Haar-$\mathcal{I}$ sets

Jarosław Swaczyna (Łódź)

In locally compact Polish groups there is a very natural σ-ideal of null sets with respect to the Haar-measure. In non-locally compact groups there is no Haar measure, however Christensen in 1972 introduced a notion of Haar-null sets which is an analogue of locally compact case. In 2013, Darji introduced a similar notion of Haar-meager sets. During my talk I will present some equivalent definition of Haar-null sets which leads us to joint generalization of those notions.

(This is joint work with Taras Banakh, Szymon Głąb and Eliza Jabłońska.)

Institute of Mathematics, Łódź University of Technology
Wólczańska 215, 93-005 Łódź, Poland
jswaczyna@wp.pl

Comparison of some trigonometric integrals

PIOTR SWOROWSKI (Bydgoszcz)

An orthogonal series,

$$f(x) = \sum_{n=1}^{\infty} a_n \varphi_n(x).$$

If, for a given orthogonal system $\{\varphi_n\}_{n=1}^{\infty}$ and given mode of convergence, the representation is always unique, there arises a question how coefficients of the series can be recovered from the sum f. Normally a solution is provided with the Fourier formulae, but this is subject to integrability of f. In the case of a trigonometric series,

$$f(x) = \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx,$$

convergence everywhere need not imply even the wide Denjoy integrability! It is so, e.g., for the series

$$\sum_{n=2}^{\infty} \frac{\sin nx}{\ln n}.$$

By a *trigonometric integral* we mean any integral with the property that the sum of every everywhere convergent trigonometric series is integrable.

Two trigonometric integrals, SCP- and AS-integrals, will be considered in connection with their consistence. Two *continuous* indefinite integrals, SCP-integral F_1 and AS-integral F_2, with the property that $F_1 - F_2$ is a singular Cantor-like function, are constructed.

(Joint work with Valentin A. Skvortsov.)

Institute of Mathematics, Kazimierz Wielki University
Pl. Weyssenhoffa 11, 85-072 Bydgoszcz, Poland
piotrus@ukw.edu.pl

Ideal convergence and matrix summability

Jacek Tryba (Gdańsk)

We examine relationship between ideal convergence and matrix summability in the realm of bounded and unbounded sequences. We present the Problem 5 from the Scottish Book [3], stated by Mazur, that can be described as *is the notion of statistical convergence of bounded sequences equivalent to some matrix summability method?*

We investigate the claims written by Mazur in the book that would lead to the negative answer to that problem and present the results of Khan and Orhan [2], which gave us a positive answer to the Problem 5 from the Scottish Book.

We also examine ideals for which Mazur's false claim written in the book holds and show when ideal convergence is equal to the intersection of some matrix summability methods. In particular, we solve a problem posed by Gogola, Mačaj and Visnyai [1].

(These results were obtained jointly with Rafał Filipów.)

References

[1] J. Gogola, M. Mačaj, T. Visnyai, *On $\mathcal{I}_c^{(q)}$-convergence*, Ann. Math. Inform., **38** (2011), 27–36.
[2] M. K. Khan, C. Orhan, *Matrix characterization of A-statistical convergence*, J. Math. Anal. Appl., **335**(1) (2007), 406–417.
[3] R. Daniel Mauldin (ed.), *The Scottish Book*, Birkhäuser/Springer, Cham, 2015,, Mathematics from the Scottish Café, Including selected papers presented at the Scottish Book Conference held at North Texas State University, Denton, Tex., May 1979.

Institute of Mathematics, Faculty of Mathematics, Physics and Informatics, University of Gdańsk
Wita Stwosza 57, 80-308 Gdańsk, Poland
jtryba@mat.ug.edu.pl

On density type topologies

RENATA WIERTELAK (Łódź)

Let $\mathbb{R}$ be the set of real numbers, $\mathbb{N}$ — the set of natural numbers. By $\lambda(A)$ we shall denote the Lebesgue measure of a measurable set A.

We shall say that a point $x_0 \in \mathbb{R}$ is a *density point* of a Lebesgue measurable set A if

$$\lim_{h\to 0^+} \frac{\lambda(A\cap[x_0-h,x_0+h])}{2h} = 1.$$

This notion was generalized in many ways. If we change the differentiation basis, then we obtain notion of $\langle s\rangle$-density topology [2], $\mathcal{J}$-density topology [3]. The most general approach (from this point of view) was presented in [6]. Namely, we say that x is an $\mathcal{S}=\{S_n\}_{n\in\mathbb{N}}$-*density point* of a measurable set A, if

$$\lim_{n\to\infty} \frac{\lambda((A-x)\cap S_n)}{\lambda(S_n)} = 1.$$

The notion of density point was transfered to the family of Baire sets. In the paper [5] the following notion was introduced. The point x is called an $\mathcal{I}$-*density point* of a Baire set A if

$$\chi_{\frac{2}{n}(A-x)\cap[-1,1]}(x) \xrightarrow[n\to\infty]{\mathcal{I}} \chi_{[-1,1]}(x).$$

Using the above notions, we can define density type topologies. I would like to present similarities and differences between them.

REFERENCES

[1] M. Csörnyei, *Density theorems revisited*, Acta Sci. Math., **64** (1998), 59–65.
[2] M. Filipczak, J. Hejduk, *On topologies associated with the Lebesgue measure*, Tatra Mt. Math. Publ., **28** (2004), 187–197.
[3] J. Hejduk, R. Wiertelak, *On the generalization of density topologies on the real line*, Math. Slovaca, **64**(5) (2014), 1267–1276.
[4] J. Hejduk, R. Wiertelak, *On the abstract density topologies generated by lower and almost lower density operators*, In: Traditional and Present-day Topics in Real Analysis, Łódź University Press, Łódź, 2013.
[5] W. Poreda, E. Wagner-Bojakowska, W. Wilczyński, *A category analogue of the density topology*, Fund. Math., **125** (1985), 167–173.
[6] F. Strobin, R. Wiertelak, *On a generalization of density topologies on the real line*, Topology Appl., **199** (2016), 1–16.

Faculty of Mathematics and Computer Science, Łódź University
Banacha 22, 90-238 Łódź, Poland
`wiertelak@math.uni.lodz.pl`

Category analogues of the density topology

Władysław Wilczyński (Łódź)

The lecture starts with a new characterization of a density point (with respect to measure). This charaterization avoids measure, only the family of sets of measure zero is important. Then the notion of an intensity point is introduced as a category analogue of a density point. It is proved that this notion is incomparable with the I-density point used earlier in numerous papers. Basic properties of the intensity topology and intensely continuous functions are studied. Many of them are similar, for example intensely continuous functions are Darboux Baire one, however both topologies are not homeomorphic.

Faculty of Mathematics and Computer Science, Łódź University
Banacha 22, 90-238 Łódź, Poland
wwil@uni.lodz.pl

On the limits of sequences of Świątkowski functions

Julia Wódka (Łódź)

We present a characterization of the uniform and pointwise limits of the sequences of Świątkowski functions.

Function f is a *Świątkowski function* ($f \in \acute{\mathcal{S}}$) if for all $a < b$ with $f(a) \neq f(b)$, there is y between $f(a)$ and $f(b)$ and $x \in (a,b) \cap \mathcal{C}(f)$ such that $f(x) = y$. Fix $A \subset \mathbb{R}$, an interval $J \subset \mathbb{R}$ and $\varepsilon > 0$. We say that a function $f\colon \mathbb{R} \to \mathbb{R}$

- satisfies the condition $S(J, A, \varepsilon)$ if for each $a, b \in J$ with $f(a) < f(b)$ there exists $x \in A \cap I(a,b)$ such that $f(x) \in (f(a) - \varepsilon, f(b) + \varepsilon)$;
- satisfies the condition $S(A, \varepsilon)$ if the union of all open intervals J for which $S(J, A, \varepsilon)$ holds is dense in $\mathbb{R}$.

Let $\mathfrak{S}$ denote the class of all functions $f \in \mathcal{Ba}$ that satisfy the condition $S(A, \varepsilon)$ for all residual sets $A \subset \mathbb{R}$ and any $\varepsilon > 0$.

Theorem. *Let $f\colon \mathbb{R} \to \mathbb{R}$.*

- *There exists a sequence $\{f_n\}$ of Świątkowski functions such that $f_n \rightrightarrows f$ if and only if f satisfies the condition $S(\mathbb{R}, \mathcal{C}(f), \varepsilon)$ for all $\varepsilon > 0$.*
- *There exists a sequence $\{f_n\}$ of Świątkowski functions such that $f_n \to f$ if and only if $f \in \mathfrak{S}$.*

References

[1] T. Natkaniec, J. Wódka, *On the pointwise limits of sequences of Świątkowski functions*, submitted.
[2] J. Wódka, *On the uniform limits of sequences of Świątkowski functions*, Lithuanian Math. J., to appear.

Institute of Mathematics, Łódź University of Technology
Wólczańska 215, 93-005 Łódź, Poland
JuliaWodka@gmail.com